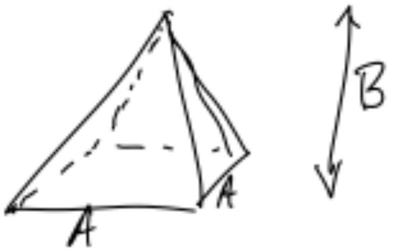
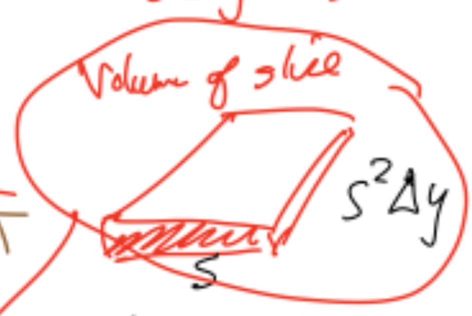
Example – Find the formula for the volume of a square pyramid, where the base of the pyramid has side length A, and the height of the pyramid is B.



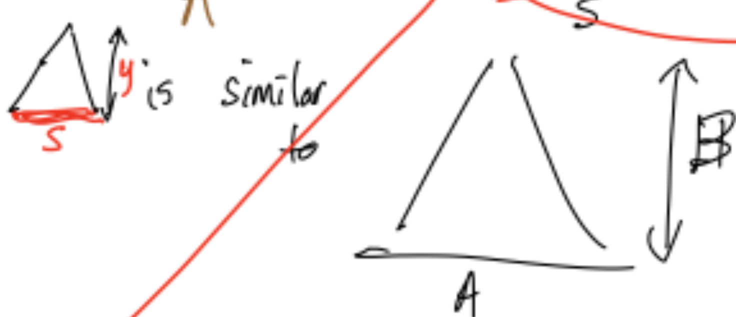
Nice way to slice – horizontal slices

Side view

$0 \le y \le B$



Similar triangles



$$\frac{S}{y} = \frac{A}{B}$$

$$S = \frac{A}{B}y$$

Volume of slice $= S^2 \Delta y$

$$= \frac{A^2}{B^2} y^2 \Delta y$$

$$\text{Total Volume} = \int (\text{Volume of slice})$$

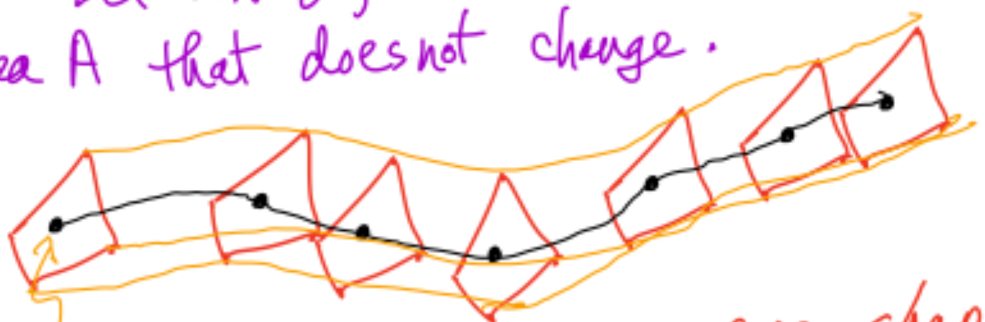
$$= \int_0^B \frac{A^2}{B^2} y^2 \, dy$$

$$= \frac{A^2}{B^2} \frac{y^3}{3} \Big|_0^B = \frac{A^2}{B^2}\frac{B^3}{3} = \boxed{\frac{A^2 B}{3}}$$

$$\boxed{\Rightarrow \text{Volume of square pyramid} = \frac{A^2 B}{3}.}$$

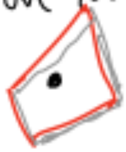
Cavalieri's Principle

Let an object have a cross-section with area A that does not change.



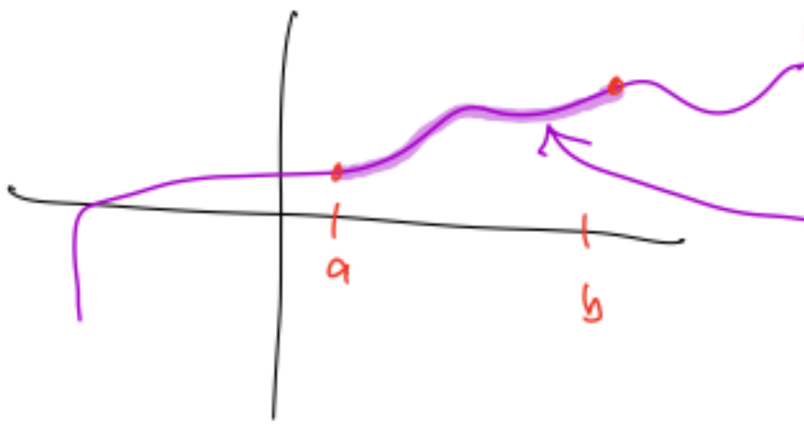
Then Volume of the shape = A · (length of curve).

Can prove this where every slice is

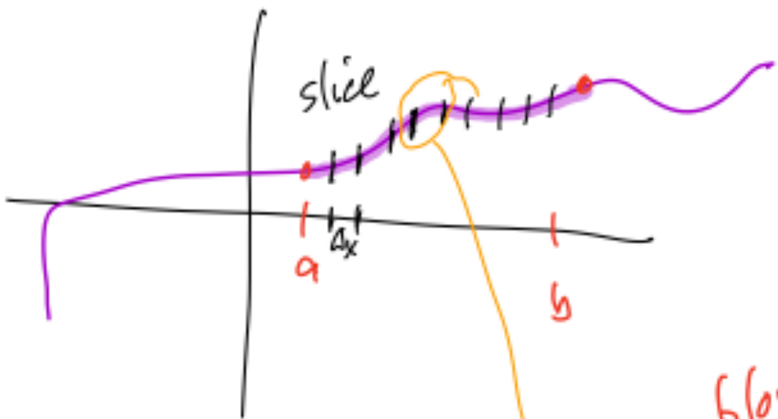


← volume of slice $= A \cdot \underbrace{\Delta s}_{\text{arclength.}}$

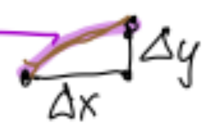
Our next geometry application of integrals: lengths of curves.



how do we compute the length?



blowup picture of one slice



$\underset{\text{of slice}}{\text{length}} \approx \sqrt{(\Delta x)^2+(\Delta y)^2}$.

$$\begin{pmatrix}\text{Total length}\\ \text{of curve}\end{pmatrix} = \int_a^b \underbrace{\sqrt{(dx)^2+(dy)^2}}_{\text{length of slice}}$$

$$\left.\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{(dx)^2\left[1+\frac{(dy)^2}{(dx)^2}\right]}\\ &= dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}\end{aligned}\right]\text{length of slice}$$

OR

$$\sqrt{(dx)^2+(dy)^2} = dy\sqrt{\left(\frac{dx}{dy}\right)^2+1}$$

Bottom line:

$$\left(\begin{array}{c}\text{Length of}\\ y=f(x)\\ \text{from } x=a\\ \text{to } x=b\end{array}\right) = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

OR

$$\left(\begin{array}{c}\text{Length of}\\ x=g(y)\\ \text{from } y=a\\ \text{to } y=b\end{array}\right) = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy$$

$\boxed{\text{Example}}$ Let $y = e^{-2x}+e^{2x}$

Find the length of this curve from $x=-1$ to $x=1$.

$$\text{Length} = \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{-1}^{1} \sqrt{1 + \left(-2e^{-2x} + 2e^{2x}\right)^2}\, dx$$

$$= \int_{-1}^{1} \sqrt{1 + 4\left(-e^{-2x} + e^{2x}\right)^2}\, dx$$

$$2 \cdot e^{-2x} \cdot e^{2x} = 2$$

$$= \int_{-1}^{1} \sqrt{1 + 4\left(e^{-4x} + e^{4x} - 2\right)}\, dx$$

$$= \int_{-1}^{1} \sqrt{-7 + 4e^{-4x} + 4e^{4x}}\, dx$$

= <u>numerically</u> compute.